Inducing π-partial characters with a given vertex

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Abstract

Let G be a solvable group. Let p be a prime and let Q be a p-subgroup of a subgroup V. Suppose $\varphi \in \mathrm{IBr}(G)$. If either $|G|$ is odd or $p = 2$, we prove that the number of Brauer characters of H inducing φ with vertex Q is at most $|\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$.

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1 Introduction

Throughout this note, G is a finite group, and $\mathrm{Irr}(G)$ is the set of irreducible characters of G. Suppose $\chi \in \mathrm{Irr}(G)$ and H is a subgroup of G. It is easy to obtain an upper bound on the number of characters in $\mathrm{Irr}(H)$ that induce χ. Let $\varphi_1, \ldots, \varphi_n \in \mathrm{Irr}(H)$ be the characters so that $\varphi_i^G = \chi$. Evaluating at 1, we obtain $\varphi_i(1) = \chi(1)/|G : H|$ for each i. By Frobenius reciprocity (Lemma 5.2 of [5]), each φ_i is a constituent of χ_H with multiplicity 1. Since there are n such characters occurring as constituents of χ_H, it follows that $n(\chi(1)/|G : H|) \leq \chi(1)$. We deduce that $n \leq |G : H|$, and we have an upper bound. If H is normal in G, this bound is obtained, and it is not particularly difficult to find nonnormal subgroups where this bound is obtained.

We now turn our attention to Brauer characters. Fix a prime p. We will write $\mathrm{IBr}(G)$ for the irreducible p-Brauer characters of G. If $\varphi \in \mathrm{IBr}(G)$, then it is easy to adapt the above proof to show that φ is induced by at most $|G : H|$ Brauer characters of H. However, associated with φ are certain p-subgroups of G called the vertex subgroups. When G is a p-solvable group, a p-subgroup Q of G is defined to be a vertex for φ if there is a subgroup U of G so that φ is induced by a Brauer character of U with p'-degree and Q is a Sylow subgroup of U. It is known that all the vertex subgroups of φ are conjugate in G. If φ is induced from $\tau \in \mathrm{IBr}(H)$, it is easy to see that a vertex for τ is a vertex for φ. Thus, H contains some vertex Q for φ. Now, different Brauer characters of H that induce φ may have vertex subgroups that are not conjugate in H but are necessarily conjugate in G. Hence, one can ask the following question: Suppose $\varphi \in \mathrm{IBr}(G)$ has vertex Q, and $Q \leq H$, how many

characters in $\mathrm{IBr}(H)$ with vertex Q induce φ? When either $|G|$ is odd or G is solvable and $p = 2$, we can obtain an upper bound for this question.

Theorem 1. *Let G be a solvable group and p a prime. Assume either $|G|$ is odd or $p = 2$. Let Q be a p-subgroup of H. If $\varphi \in \mathrm{IBr}(G)$, then the number of Brauer characters of H with vertex Q that induce φ is at most $|\mathrm{N}_G(Q) : \mathrm{N}_H(Q)|$.*

At this time, we are not able to determine whether or not this theorem is true if we loosen the hypothesis that either $|G|$ is odd or $p = 2$. In other words, is the conclusion still true if G is a solvable group of even order and p is an odd prime. This result was motivated by our work with J. P. Cossey. If we could prove the conclusion of Theorem 1 when p is odd, then we would be able to prove J. P. Cossey's conjecture that the number of lifts of a Brauer character is bounded by the index of a vertex subgroup in the vertex subgroup when p is odd. Our argument can be found in the preprint [1].

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2 Results

We will in the more general setting of irreducible π-partial characters of a π-separable group G. We here briefly mention that if π is a set of primes and G is a π-separable group, one can define (see [10] for more details) a set of class functions $\mathrm{I}_\pi(G)$ from the set G^o (which consists of the elements of G whose order is divisible by only the primes in π) to $\mathbf{C}$ that plays the role of $\mathrm{IBr}(G)$, and in fact $\mathrm{I}_\pi(G) = \mathrm{IBr}(G)$ if $\pi = \{p'\}$, the complement of the prime p.

We start by considering vertices in Clifford correspondence (see Proposition 3.2 of [9]). Let G be a π-separable group. Let N be a normal subgroup of G. Fix $\varphi \in \mathrm{I}_\pi(G)$. If $\alpha \in \mathrm{I}_\pi(N)$ is a constituent of φ_N, then we write G_α for the stabilizer of α in G, and we write φ_α for the Clifford correspondent of φ with respect to α. In particular, the vertices of the Clifford correspondent form an orbit under the action of the normalizer of a particular vertex.

Lemma 2. *Let G be a π-separable group. Let N be a normal subgroup of G. Suppose that $\alpha \in \mathrm{I}_\pi(N)$. Let $\varphi \in \mathrm{I}_\pi(G)$ and $\hat{\varphi} \in \mathrm{I}_\pi(G_\alpha)$ so that $\hat{\varphi}^G = \varphi$. Suppose that Q is a vertex for $\hat{\varphi}$. Then Q is a vertex is $\hat{\varphi}^g$ if and only if there exists $n \in \mathrm{N}_G(Q)$ so that $G_\alpha g = G_\alpha n$.*

Proof. We first suppose that there exists $n \in \mathrm{N}_G(Q)$ so that $G_\alpha g = G_\alpha n$. Thus, $g = tn$ for some $t \in G_\alpha$. We see that $\hat{\varphi}^g = \hat{\varphi}^{tn} = \hat{\varphi}^n$. We see that $Q = Q^n$ is a vertex for $\hat{\varphi}^n = \hat{\varphi}^g$.

Conversely, suppose that Q is a vertex for $\hat{\varphi}^g$. Then $Q^{g^{-1}}$ is a vertex for $\hat{\varphi}$. Since Q is also a vertex for $\hat{\varphi}$, we have $Q^{g^{-1}} = Q^t$ for some $t \in G_\alpha$. It follows that $Q = Q^{tg}$, and so, $tg \in \mathrm{N}_G(Q)$. This implies that $tg = n$ for some $n \in \mathrm{N}_G(Q)$. This implies that $n \in G_\alpha g$, and we conclude that $G_\alpha n = G_\alpha g$. □

We continue to work in the context of the Clifford correspondence. In this case, we can get an exact count of the number of partial characters in N whose Clifford correspondent has vertex Q.

Corollary 3. *Let G be a π-separable group. Let N be a normal subgroup of G, let $\varphi \in \mathrm{I}_\pi(G)$ have vertex Q, and suppose that β is an irreducible constituent of φ_N so that φ_β has vertex Q. Then* $|\{\alpha \in \mathrm{I}_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\}| = |\mathrm{N}_G(Q) : \mathrm{N}_{G_\beta}(Q)|$.

Proof. By Lemma 2, we see that φ_α has Q as a vertex if and only if $\alpha = \beta^g$ where $g \in G$ and $g \in G_\beta n$ for some $n \in \mathrm{N}_G(Q)$. Finally, we observe that $G_\beta n_1 = G_\beta n_2$ if and only if $\mathrm{N}_{G_\beta}(Q)n_1 = \mathrm{N}_{G_\beta}(Q)n_2$ for $n_1, n_2 \in \mathrm{N}_G(Q)$. We have $|\{\alpha \in \mathrm{I}_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\}| = |\{G_\beta n \mid n \in \mathrm{N}_G(Q)\}| = |\mathrm{N}_G(Q) : \mathrm{N}_{G_\beta}(Q)|$. □

We now look at the conditions of a minimal counterexample. For this we need to review and develop more notation. We make use of the canonical set of π-lifts, $\mathrm{B}_\pi(G)$, that was defined in [8] by Isaacs. In other words, $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ and the map $\chi \mapsto \chi^o$ is a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$. Closely related to this set is the subnormal nucleus which also was defined in [8]. To define the subnormal nucleus, we need the π-special characters. Let G be a π-separable group. A character $\chi \in \mathrm{Irr}(G)$ is π-special if $\chi(1)$ is a π-number and for every subnormal group M of G, the irreducible constituents of χ_M have determinants that have π-order. Many of the basic results of π-special characters can be found in Section 40 of [2] and Chapter VI of [13]. One result that is proved is that if α is π-special and β is π'-special, then $\alpha\beta$ is necessarily irreducible. We say that χ is **factored** if $\chi = \alpha\beta$ where α is π-special and β is π'-special. We also note that if $\chi \in \mathrm{B}_\pi(G)$ and N is normal in G, then the irreducible constituents of χ_N lie in $\mathrm{B}_\pi(N)$.

If $\chi \in \mathrm{Irr}(G)$, Isaacs constructs the subnormal vertex as follows. Let M be maximal so that M is subnormal in G and the irreducible constituents of χ_M are factored. Let μ be an irreducible constituent of χ_M and let T be the stabilizer of (M, μ) in G. Isaacs proved in [8] that there is a Clifford theorem for T. In other words, there is a unique character $\tau \in \mathrm{Irr}(T \mid \mu)$ so that $\tau^G = \chi$. He also proved that (M, μ) is unique up to conjugacy, and so, (T, τ) is unique up to conjugacy. If $T = G$, then χ is π-factored and we take (G, χ) to be the subnormal nucleus of χ. If $T < G$, then inductively, the subnormal nucleus for τ is the subnormal nucleus for χ. We write (W, γ) for the subnormal nucleus of χ, and Isaacs showed that $\gamma^G = \chi$, γ is factored, and (W, γ) is unique up to conjugacy. A character $\chi \in \mathrm{Irr}(G)$ is in $\mathrm{B}_\pi(G)$ if and only if the character of its nucleus is π-special.

If Q is a π'-subgroup of G, then we use $\mathrm{I}_\pi(G \mid Q)$ to denote the π-partial characters in $\mathrm{I}_\pi(G)$ that have vertex Q. If $\varphi \in \mathrm{I}_\pi(G)$ and $V \leq G$, then we write $\mathrm{I}_\varphi(V \mid Q) = \{\eta \in \mathrm{I}_\pi(V \mid Q) \mid \eta^G = \varphi\}$. We now find details about properties of a minimal counterexample. We will see that a counterexample cannot occur when either $|G|$ is odd or 2 is not in π. Our goal is find enough information so that we can either find a contradiction or build an example when $|G|$ is even and $2 \in \pi$.

Theorem 4. *Let G be a solvable group. Assume $\varphi \in \mathrm{I}_\pi(G)$ has vertex Q, let V be a subgroup of G, and let N be the core of V in G. If G and V are chosen so that $|G| + |G : V|$ is minimal subject to the condition that $|\mathrm{I}_\varphi(V \mid Q)| > |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$, then the following are true:*

1. *V is a nonnormal maximal subgroup of G,*

2. *$|G : V|$ is a power of 2,*

3. *$2 \in \pi$,*

4. $Q \le V$*,*

5. $\varphi_N = a\alpha$ *for some* $\alpha \in \mathrm{IBr}(N)$*,*

6. $\alpha(1)$ *is a* π*-number,*

7. if K *is normal in* G *so that* K/N *is a chief factor for* G*, then* α *is fully ramified with respect to* K/N*.*

Proof. If either $V = G$ or $\mathrm{I}_\varphi(V \mid Q)$ is empty, then $|\mathrm{I}_\varphi(V \mid Q)| \le |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$ contradicting the hypotheses. Thus, $V < G$ and $\mathrm{I}_\varphi(V \mid Q)$ is not empty, and so, $Q \le V$ and there exist characters in $\mathrm{I}_\pi(V)$ that induce φ and have vertex Q.

We begin by showing that V is a maximal subgroup. Suppose that $V < M < G$ for some subgroup M. Let $\mathrm{I}_\varphi(M \mid Q) = \{\eta_1, \ldots, \eta_m\}$. Using minimality, we have $m = |\mathrm{I}_\varphi(M \mid Q)| \le |\mathrm{N}_G(Q) : \mathrm{N}_M(Q)|$. Suppose that $\zeta \in \mathrm{I}_\varphi(V \mid Q)$, then $\zeta^M \in \mathrm{I}_\pi(M)$ and ζ^M has Q as a vertex. Since $(\zeta^M)^G = \zeta^G = \varphi$, we see that $\zeta^M \in \mathrm{I}_\varphi(M \mid Q)$. It follows that $\zeta^M = \eta_i$ for some i. We conclude that $|\mathrm{I}_\varphi(V \mid Q)| \le \sum_{i=1}^m |\mathrm{I}_{\eta_i}(V \mid Q)|$. Since this contradicts our hypothesis, we obtain $|\mathrm{I}_{\eta_i}(V \mid Q)| \le |\mathrm{N}_M(Q) : \mathrm{N}_V(Q)|$. We deduce that

$$|\mathrm{I}_\varphi(V \mid Q)| \le m|\mathrm{N}_M(Q) : \mathrm{N}_V(Q)| \le |\mathrm{N}_G(Q) : \mathrm{N}_M(Q)||\mathrm{N}_M(Q) : \mathrm{N}_V(Q)| = |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|.$$

Since this violates the hypotheses, V is maximal in G.

If V is normal in G, then either φ is induced from V or φ restricts irreducibly to V. If φ is induced from V, then we can apply Corollary 3 to see that $|\mathrm{I}_\varphi(V \mid Q)| \le |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$ in violation of the hypotheses. If φ restricts irreducibly, then it cannot be induced from V, and we have seen that this is also a contradiction. We conclude that V is not normal in G.

Suppose $\alpha \in \mathrm{I}_\pi(N)$ is a constituent of φ_N. We use $\varphi_\alpha \in \mathrm{I}_\pi(G_\alpha \mid \alpha)$ to denote the Clifford correspondent for φ with respect to α (see Proposition 3.2 of [9] again). Write $\{\alpha \in \mathrm{I}_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\} = \{\alpha_1, \ldots, \alpha_k\}$, and let $\varphi_i = \varphi_{\alpha_i}$ and $G_i = G_{\alpha_i}$. By Lemma 3, we know that $k = |\mathrm{N}_G(Q) : \mathrm{N}_{G_i}(Q)|$.

Suppose $\eta \in \mathrm{I}_\varphi(V \mid Q)$. Denote $\{\beta \in \mathrm{I}_\pi(N) \mid \eta_\beta \text{ has vertex } Q\} = \{\beta_1, \ldots, \beta_l\}$, and let $\eta_j = \eta_{\beta_j}$ and $V_j = V_{\beta_j}$. By Lemma 3, $l = |\mathrm{N}_V(Q) : \mathrm{N}_{V_i}(Q)|$.

We see that $(\eta_j)^G = ((\eta_j)^V)^G = \eta^G = \varphi$. This implies that $(\eta_j)^{G_{\beta_j}}$ is irreducible and has vertex Q. It follows that $\beta_j = \alpha_{i_j}$ for some i_j. We obtain $G_{\beta_j} = G_{i_j}$ and $(\beta_j)^{G_{i_j}} = \alpha_{i_j}$. Observe that $V_j = G_{i_j} \cap V$, and we denote this subgroup by $V^*_{i_j}$.

Now, we assume that $k > 1$, and we start to count. We see that $\eta \in \mathrm{I}_\varphi(G \mid Q)$ is induced by $|\mathrm{N}_V(Q) : \mathrm{N}_{V^*_{i_j}}(Q)|$ partial characters in $\bigcup \mathrm{I}_{\varphi_i}(V^*_i \mid Q)$. Because $G_i < G$, we may use minimality of $|G| + |G : V|$ to deduce $|\mathrm{I}_{\varphi_i}(V^*_i \mid Q)| \le |\mathrm{N}_{G_i}(Q) : \mathrm{N}_{V^*_i}(Q)|$. We compute

$$|\mathrm{I}_\varphi(V \mid Q)| = \sum_{i=1}^{k} \frac{1}{|\mathrm{N}_V(Q) : \mathrm{N}_{V^*_i}(Q)|} |\mathrm{I}_{\varphi_i}(V^*_i \mid Q)| \le \sum_{i=1}^{k} \frac{1}{|\mathrm{N}_V(Q) : \mathrm{N}_{V^*_i}(Q)|} |\mathrm{N}_{G_i}(Q) : \mathrm{N}_{V^*_i}(Q)|.$$

We determine that

$$\frac{1}{|\mathrm{N}_V(Q) : \mathrm{N}_{V^*_i}(Q)|} |\mathrm{N}_{G_i}(Q) : \mathrm{N}_{V^*_i}(Q)| = \frac{|\mathrm{N}_{G_i}(Q)|}{|\mathrm{N}_V(Q)|},$$

for each i. Notice that $|\mathrm{N}_{G_i}(Q)| = |\mathrm{N}_{G_1}(Q)$ for all i and $k = |\mathrm{N}_G(Q) : \mathrm{N}_{G_1}(Q)$. This yields

$$|\mathrm{I}_\varphi(V \mid Q)| \leq \sum_{i=1} k \frac{|\mathrm{N}_{G_1}(Q)}{\mathrm{N}_V(Q)} = \frac{|\mathrm{N}_G(Q) : \mathrm{N}_{G_1}(Q)||\mathrm{N}_{G_1}(Q)|}{|\mathrm{N}_V(Q)|} = |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|.$$

This contradicts the hypothesis. We deduce that $k = 1$, and α is invariant in G.

Set $\alpha = \alpha_1$, and let α^* be the character in $\mathrm{B}_\pi(N)$ satisfying $(\alpha^*)^o = \alpha$. Write $(W, \hat{\alpha})$ for the nucleus of α^*, and take T to be the stabilizer of $(W, \hat{\alpha})$ in G. By Lemma 2.3 of [11], there is a unique character $\hat{\varphi} \in \mathrm{I}(T \mid \hat{\alpha})$ so that $\hat{\varphi}^G = \varphi$ and Q is a vertex for $\hat{\varphi}$. Similarly, if $\eta \in \mathrm{I}_\varphi(V \mid Q)$, then there is a unique character $\hat{\eta} \in \mathrm{I}(T \cap V \mid \hat{\alpha})$ so that $\hat{\eta}^V = \eta$ and Q is a vertex for $\hat{\eta}$. Observe that $\hat{\eta}^T \in \mathrm{I}(T \mid \hat{\alpha})$ and induces φ, so $\hat{\eta}^T = \hat{\varphi}$. It follows that $|\mathrm{I}_\varphi(V \mid Q)| = |\mathrm{I}_{\hat{\varphi}}(T \cap V \mid Q)|$. If $T < G$, then we can use the minimality of $|G| + |G : V|$ to see that $|\mathrm{I}_{\hat{\varphi}}(T \cap V \mid Q)| \leq |\mathrm{N}_T(Q) : \mathrm{N}_{V \cap T}(Q)|$. By the diamond lemma, we have $|\mathrm{N}_T(Q) : \mathrm{N}_{V \cap T}(Q)| = |\mathrm{N}_T(Q) : V \cap \mathrm{N}_T(Q)| \leq |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$. This contradicts the hypotheses, and so $T = G$.

We now have that $(W, \hat{\alpha})$ is G-invariant. By the construction of the subnormal, this implies that $W = N$. Since $\alpha^* \in \mathrm{B}_\pi(N)$, the nucleus for α^* has a character that is π-special. Thus, $\hat{\alpha}$ is π-special, and since $W = N$, we see that $\hat{\alpha} = \alpha^*$. In particular, $\hat{\alpha}$ is π-special. We deduce that $\alpha(1)$ is a π-number.

Take K normal in G so that K/N is a chief factor for G. This is the point where we use the fact that G is solvable to see that $G = VK$ and $V \cap K = N$ where K/N is an elementary abelian p-group for some prime p. (This is the only place we use the hypothesis that G is solvable in place of G being π-separable.) Let L/K be a chief factor for G. We know that $(|L : K|, |K : N|) = 1$ and $\mathbf{C}_{L \cap V/N}(K/N)$. (See Lemma 5.1 of [12] for a proof of this.) By Problem 6.12 of [5], either α^* extends to K or α^* is fully-ramified with respect to K/N. .

Suppose first that α^* extends to K. Notice that multiplication by $\mathrm{Irr}(K/L)$ is a transitive action on the irreducible constituents of $(\alpha^*)^K$. Also, $(V \cap K)/L$ acts on compatibly on the irreducible constituents of $(\alpha^*)^K$ and on $\mathrm{Irr}(K/L)$ where the action on $\mathrm{Irr}(K/L)$ is coprime. We can use Glauberman's lemma (Lemma 13.8 of [5]) to see that α^* has a $V \cap L$-invariant extension. The corollary to Glauberman's lemma (Corollary 13.9 of [5]) can be applied to see that α^* has a unique $V \cap L$-invariant extension δ. Since V permutes the $V \cap L$-extensions of α^*, it follows that δ is V-invariant. We now use Corollary 4.2 of [8] to see that restriction is a bijection from $\mathrm{Irr}(G \mid \beta)$ to $\mathrm{Irr}(V \mid \alpha^*)$.

Let $\eta \in \mathrm{I}_\varphi(V \mid Q)$ so that $\eta^G = \varphi$. We can find $\eta^* \in \mathrm{B}_\pi(V)$ so that $(\eta^*)^o = \eta$. Since $(\eta^{*G})^o = (\eta^{*o})^G = \eta^G = \varphi \in \mathrm{IBr}(G)$, we see that η^G is irreducible. On the other hand, $(\eta^{*o})_N = (\eta_N)^o = b\alpha$ for some integer b. Since the irreducible constituents of ${\eta^*}_N$ lie in $\mathrm{B}_\pi(N)$, we deduce that $\eta^* \in \mathrm{Irr}(V \mid \alpha^*)$. But we saw that this implies that η^* extends to G. Since $V < G$, it is not possible for η^* to both extend to G and induce irreducibly. Therefore, we have a contradiction. We see that α^* (and hence, α) is fully ramified with respect to K/N. Notice that if p is not in π, then Corollary 6.28 of [5] applies and α^* extends to K. Therefore, $p \in \pi$.

We suppose that p is odd, and we work for a contradiction. Since α^* is fully-ramified with respect to K/N and $|K : N|$ has odd order, main theorem of [7] implies that no character in $\mathrm{Irr}(V \mid \alpha)$ induces irreducibly to G. (A stronger theorem is proved in [4].) As in the previous paragraph, this implies that φ is not induced from V which contradicts the assumption that

$\mathrm{I}_\varphi(V \mid Q)$ is not empty. (This strongly uses the fact that p is odd. When $p = 2$, it is tempting to try use the correspondence in [6], but that correspondence does not preclude inducing characters in $\mathrm{Irr}(G \mid \alpha)$ from V. In fact, $\mathrm{GL}_2(3)$ is an example where this occurs.) We conclude that $p = 2$. Since $|G : V| = |K : N|$, we see that $|G : V|$ is a power of 2. This proves the theorem. $\square$

As a corollary, we obtain Theorem 1 stated for π-partial characters.

Corollary 5. *Let G be a solvable group. Assume either $|G|$ is odd or $2 \notin \pi$. Let Q be a π'-subgroup of G and suppose that $Q \le V$. If $\varphi \in \mathrm{I}_\pi(G)$, then $|\mathrm{I}_\varphi(V \mid Q)| \le |\mathrm{N}_G(Q) : \mathrm{N}_V(Q)|$.*

Proof. We suppose the result is not true. Let G be a counterexample with $|G| + |G : V|$ as in Theorem 4. By that result, we have that $|G : V|$ is a nontrivial power of 2 which is a contradiction if $|G|$ is odd. We also have $2 \in \pi$ which is a contradiction to $2 \notin \pi$. This proves the corollary. $\square$

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